Effects of the relative scales between particles added and toppled each time on the distribution of avalanche size in Abelian sandpiles

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We study the distribution function of avalanche size, $P_x(s)$, for a general Abelian sandpile as a function of the number of particles x added each time. An exact piecewise linear relation between various P_x is derived provided that the particles are added to one and only one site each time. This is then verified by numerical simulation. We find that $P_x(s)$ depends, but maybe not very strongly in general, on x. Thus x can be regarded as a fine-tune parameter for the Abelian sandpiles.

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The idea of self-organized criticality was introduced by Bak, Tang, and Wiesenfeld, using sandpile as an example, that some complex systems may organize according to their own dynamics into states where there are no characteristic length and time scales. Thus scaling behavior in some of their physical quantities is observed [1]. There is a particular type of sandpile model, the Abelian models, that are interesting because of the rich mathematical structures found [2]. The general rule of the game, for conventional Abelian sandpile models can be summarized as follows [1-4]: a unit number of particles is added to a site according to a certain underlying probability distribution, which is taken to be uniform by most authors. Whenever the number of particles in a site is greater than a certain triggering threshold, rearrangement of particles called toppling will occur in the next time step, which is given by [2]

$$h_j \longrightarrow h_j - \Delta_{ij}, \ \forall\ j \qquad (1)$$

whenever h_i, the number of particles in site i, is greater than the triggering threshold. Without lost of generality, we can simply rescale the triggering threshold at each site to zero. The total number of toppling occuring when a particle is introduced into the system defines the avalanche size. Upon repeated introduction of particles and their subsequent relaxation via toppling, a stationary state is reached and the distribution function of avalanche size $P(s)$ can at least be computed numerically. Whenever a local particle conservation law exists in the process of toppling, that is when particles can only be allowed to dissipate at the system boundary, scaling behavior in $P(s)$ is observed [1,5].

The number of particles x added to the pile all at each instant of time is proportional to the rate of introduction of stress onto the system between two successive avalanches. Similarly the size of the toppling matrix Δ represents the rate of stress releasing for each single toppling: if Δ is multiplied by a constant $\lambda > 1$, then in general, it takes approximately $1/\lambda$ times the original number of topplings to bring the system back to stability. Thus it is interesting to know how the relative size between the number of particles added and the number of particles involved in toppling each time in each site affects the distribution of avalanches $P(s)$. Without loss of generality, we shall keep the toppling matrix Δ fixed and vary the number of particles added x only. In addition, we shall allow the number of particles in each site h_i to be a real number, so that we can vary x continuously. The result, however, is also valid even if h_i can take on only integral values. Discussion of similar continuous local height models can be found elsewhere [6,7].

In this letter, we are going to prove a piecewise linear relationship between various $P_x(s)$ provided that particles are introduced to one and only one site each time. This relation is then verified by numerical simulation on the two-dimensional Abelian sandpile with open boundary conditions. The result suggests that x can alter the avalanche distribution, although the distribution itself is not in general very sensitive to x. Still, x can be regarded as a fine-tuning parameter of the system.

Various authors have shown that the recurrence phase-space volume for Abelian sandpiles equals $\det \Delta$ [2,6–8]. If we regard every system configuration as a point in $\mathbb{R}^n$, the recurrence phase space, E, is therefore a subset of $\mathbb{R}^n$ where n is the total number of sites in the system. In fact, we can partition $\mathbb{R}^n$ in the following way [7,8]:

$$\mathbb{R}^n = \bigcup_{k_1,\cdots,k_n \in \mathbb{Z}} T_{k_1,\cdots,k_n}[E], \qquad (2)$$

where $T_{k_1,\cdots,k_n}(y) = y + \sum_{i=1}^n k_i \Delta_i$ with Δ_i being the ith row vector of the toppling matrix and $\mathbb{Z}$ is the set of all integers. Moreover, for any point $y \in T_{k_1,\cdots,k_n}[E]$ with all $k_i \geq 0$, it requires exactly $\sum_{i=1}^n k_i$ times of toppling to bring it back to stability [7]. Similar results for the discrete-integral case can be found elsewhere [9]. The introduction of particles into the system can be regarded as a translation $A_a[E] = \{a + y : y \in E\}$ where the ith coordinate component of a is the number of particles added to site i, and the probability that an avalanche of size s results is given by

$$P(s|a) = \sum_{\sum k_i = s} \mu(A_a[E] \cap T_{k_1,\cdots,k_n}[E]) \Big/ \mu(E), \qquad (3)$$

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where μ denotes the usual n-dimensional Lebesgue measure and the sum is over all non-negative k_i with $\sum_{i=1}^{n} k_i = s$. Whenever there is a change in the stress introduction rate as reflected by multiplying a by a positive constant, $P(s|a)$ changes because the "volumes" of $A_a[E]$ in different parts of the partition of $\mathbb{R}^n$ by $T_{k_1,\cdots,k_n}[E]$ change. Thus the avalanche distribution depends not only on how the particles are added, but also on the shape of the recurrence phase space E. In particular, if the particle is added to a particular site only (say an arbitrary but fixed site j), then $a_j \equiv x$ is the only non-vanishing component of a. And from now on, $P(s|x,j)$ is used to denote the distribution of avalanches with exactly x particles being added to site j.

It is trivial that $P(s|0,j) = \delta_{s,0}$ for all j, where $\delta_{s,0}$ is the Kronecker delta. The situation is much more complicated for $P(s|\epsilon,j)$ for some sufficiently small $\epsilon > 0$. We may write Eq. (3) as

$$P(s|\epsilon,j) = \alpha P(s|0,j) + (1-\alpha)\tilde{P}(s), \quad (4)$$

where $\alpha = \alpha(\epsilon)$ is the fractional "volume" of E with $A_{x,j}[E]$ also in E, and $\tilde{P}(s)$ denotes the contribution of $P(s|\epsilon,j)$ from the rest of the elements in E. There are those that have become unstable at least once upon the addition of particle to site j. Now we are going to show that $\tilde{P}$ is independent of ϵ provided that it is sufficiently small.

A direct consequence of Theorem 2 in [7] tells us that the boundary of E of any Abelian sandpile is a (finite) union of (bounded) sections of hyperplanes with normals parallel to the coordinate axes. In fact, E is made up of a finite union of mutually disjoint n-dimensional rectangular blocks, say,

$$E = \bigcup_i (a_{i1}, b_{i1}] \times \cdots \times (a_{in}, b_{in}]. \quad (5)$$

Note that the action of $A_{x,j}$ on E simply translates the rectangular blocks along the j axis for any $x \geq 0$. Then for any $0 < \epsilon \leq \min_i (b_{ij} - a_{ij}) \equiv M$, the "volume" of $A_{\epsilon,j}[E] \cap T_{k_1,\cdots,k_n}[E]$ for all $k_i \in \mathcal{N}$ but not all zero is proportional to ϵ. So let us write $\mu\left(A_{\epsilon,j}[E] \cap T_{k_1,\cdots,k_n}[E]\right) = \zeta_{k_1,\cdots,k_n}\epsilon$ for some constant $\zeta_{k_1,\cdots,k_n} \geq 0$. From Eq. (3), we have

$$\mu(E)\,P(s|\epsilon,j) = (1-K\epsilon)\delta_{s,0} + \sum_{\sum k_i = s'} \zeta_{k_1,\cdots,k_n}\epsilon\delta_{s,s'}, \quad (6)$$

where $K = \sum_{k_1,\cdots,k_n} \zeta_{k_1,\cdots,k_n}$. Compared with Eq. (4), it is clear that $\tilde{P}(s)$ is independent of ϵ. The above argument shows also that α depends linearly on ϵ.

When $\epsilon > M$, the situation is completely different because part of the $A_{\epsilon,j}[E]$ may enter into a new rectangular block or leave an old one. This is precisely the time when a large change in $P(s)$ may begin and M can be regarded as a critical value in this respect. If we arrange the critical values of particles added onto site j in ascending order, say $0 = M_1 < M_2 < \cdots < M_k < \cdots$, then using the same idea as above, it is not difficult to see that for any $M_{k-1} < x \leq M_k$, we have

$$P(s|x,j) = \sum_{i=0}^{k} \alpha_i \check{P}_i(s) \quad (7)$$

for some $\alpha_i \geq 0$ with $\sum_{i=0}^{k} \alpha_i = 1$, and $\check{P}_i(s)$ is independent of x for each i. Here α_i depends linearly on x. By substituting x by $M_1, \ldots, M_k$ into Eq. (7), we can rewrite it into a more useful form:

$$P(s|x,j) = \sum_{i=0}^{k} \beta_i P(s|M_i,j) \quad (8)$$

for some $\beta_i \geq 0$, which depends linearly on x, with $\sum_{i=0}^{k} \beta_i = 1$. By putting $x = M_{k-1}$ and $x = M_k$ onto Eq. (8), we have the following important linear relationship:

$$P(s|x,j) = \frac{1}{M_k - M_{k-1}}\left[(M_k - x)\,P(s|M_{k-1},j) + (x - M_{k-1})\,P(s|M_k,j)\right], \quad (9)$$

whenever $M_{k-1} \leq x \leq M_k$. As j is arbitrary and the number of sites, and hence the number of ways that particles are introduced into the system, is finite, the above result holds also for any distribution of particle addition method, uniform or not, as long as particles are added to one and only one site each time. And Eq. (9) can be formally generalized to

$$P_x(s) = \frac{1}{M_k - M_{k-1}}\left[(M_k - x)\,P_{M_{k-1}}(s) + (x - M_{k-1})\,P_{M_k}(s)\right], \quad (10)$$

whenever $M_{k-1} \leq x \leq M_k$, where P_x denotes the avalanche size distribution when x particles are added each time to one and only one site, following a prescribed random particle addition method (uniformly over all the sites or not). Thus we can construct $P_x(s)$ for any $x \geq 0$ by knowing the distributions of avalanche at all the critical points together with the values of the critical points, as long as the particles are introduced into one and only one site of the system each time. Let us consider the one-dimensional asymmetric sandpile as an example: the toppling matrix is given by

$$\Delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ -1, & \text{if } i+1 = j \\ 0, & \text{otherwise.} \end{cases} \quad (11)$$

Upon a uniform and random particle addition, direct calculation shows that $E = (-1,0]^n$ and

$$P_x(s) = \begin{cases} \frac{[1-\mathrm{frac}(x)]}{n}, & \text{if } s = k\ \mathrm{int}(x),\ k = 1,\ldots,n \\ \frac{\mathrm{frac}(x)}{n}, & \text{if } s = k\ [\mathrm{int}(x)+1],\ k = 1,\ldots,n \\ 0, & \text{otherwise,} \end{cases} \quad (12)$$

where n is the number of sites of the system, and $\text{int}(x)$ and $\text{frac}(x)$ denote the integral and fractional parts of x, respectively. Equation (10) is therefore satisfied with $M_i = i - 1$ for all $i > 0$. So each time when x is just slightly greater than an integral value, there is a sudden change in the distribution of avalanche sizes (although the recurrence phase space E remains unchanged).

This is also true for other less trivial Abelian sandpiles. We illustrate it by the avalanche size distribution of a two-dimensional symmetric pile with open boundary condition; the toppling matrix is given by [2]

$$\Delta_{i,j;k,l} = \begin{cases} 4, & \text{if } i = k, j = l \\ -1, & \text{if } i = k, |j-l| = 1 \\ -1, & \text{if } j = l, |i-k| = 1 \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

for $i, j, k, l = 1, \ldots, n$. Under a uniform and random particle addition scheme in a 50×50 grid, the avalanche size distributions $P_x(s)$ for various values of x are shown in Fig. 1, which are obtained by numerical simulation because no analytic form is known to date. The graphs shows that P_x certainly depends on x although not in a very sensitive way as compared to the one-dimensional model above. Nevertheless it is clear that we can change the distribution by altering the value of x. In this respect, x can be regarded as a fine-tune parameter of the system similar to the role of the underlying distribution of the particle addition function as discussed elsewhere [10]. The graphs shows that the range of s for which the scaling relationship $P_x(s) \approx s^{-\gamma}$ holds drops from about three decades for $x = 1$ to about two decades for $x = 4$. Values of γ for a number of different x, which are obtained by measuring the slopes of the regions of the curves where the scaling relationship holds, are listed in Table I showing its weak dependence on x. Unlike P_x, Eq. (10) tells us that $\gamma(s)$ changes smoothly as s varies across the thresholds M_k.

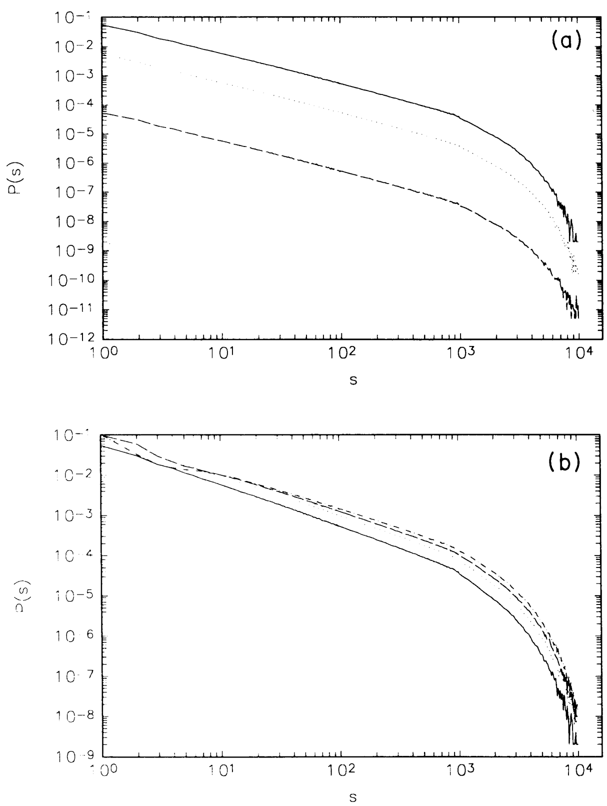

FIG. 1. Log-log plot of s against $P_x(s)$ for various x in a 50×50 grid. Note that values of $P_x(0)$ are not shown due to the singularity of the log of zero. In (a), curves with x = 1, 0.1, and 0.001 are shown in solid, dotted, and dashed lines, respectively. In (b), curves with x = 1, 2, 3, and 4 are shown in solid, dotted, dashed, and dash-dotted lines, respectively. The distribution is obtained by applying the particle addition operations ranging from 2×10^9 times for $x = 0.001$ to 4×10^6 times for $x = 4$. The fluctuations in large s are due to insufficient data.

TABLE I. Scaling exponents as a function of x.

x	γ
1.0	1.01±0.01
2.0	0.99±0.01
3.0	0.96±0.01
4.0	0.93±0.02

Because the recurrence phase space E, in this case, is made up of a finite union of hypercubes of unit length on each side, the critical values $M_i = i - 1$ for all i. Table II shows the error between the actual numerically simulated P_x and the interpolated one, thus verifying the validity of Eq. (10).

In conclusion, we have derived a piecewise linear relationship between various avalanche size distributions provided that particles are introduced to the system at one and only one site each time, which is verified by numerical simulation. Such a linear relationship is remarkable and is not expected at the first sight in this kind of complicated system. Besides, we show that x (and hence a distribution of various allowed values of x) can act as a fine tuning parameter for the Abelian sandpiles, which indicates the importance of the relative scales between stress introduction rate and the local stress releasing rate on $P(s)$. It is interesting to extend our work to both the non-Abelian sandpiles and the case where parti-

TABLE II. Errors between the simulated and the exact results of $P_x(s)$ for various x.

x	Error[a]	Iterations[b]
0.1	0.07±0.09	6×10^7
1.5	0.17±0.20	5×10^6
2.5	0.19±0.20	4×10^6
3.9	0.30±0.40	1×10^6

[a]The mean square error between the simulated and the exact result in log-log domain.

[b]Number of times where particles are introduced into the system during the simulation in order to obtain P_x, and hence γ.

cles are introduced into more than one site each time. In fact, the "local limited model" introduced by Kadanoff *et al.* and intensively studied by Chhabra *et al.* recently [4] suggests that the relative scale of the toppling matrix (and hence the number of particles added to the pile each time) is also important for noncommutative models, at least the one-dimensional ones. Work along these two lines is in progress.

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[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988).
[2] D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990); D. Dhar and S. N. Majumdar, J. Phys. A **23**, 4333 (1990); M. Creutz, Comput. Phys. **5**, 198 (1991).
[3] B. McNamara and K. Wiesenfeld, Phys. Rev. A **41**, 1867 (1990); S. C. Lee, N. Y. Liang, and W. J. Tzeng, Phys. Rev. Lett. **67**, 1479 (1991).
[4] L. P. Kadanoff, S. R. Nagel, L. Wu, and S. Zhou, Phys. Rev. A **39**, 6524 (1989); A. B. Chhabra *et al.*, Phys. Rev. E **47**, 3099 (1993).
[5] H. J. Jensen, K. Christensen, and H. C. Fogedby, Phys. Rev. B **40**, 7425 (1990); K. Christensen, H. C. Fogedby, and H. J. Jensen, J. Stat. Phys. **63**, 653 (1991); S. S. Manna, Physica A **179**, 249 (1991); H. F. Chau and K. S. Cheng, Phys. Rev. A **46**, 2981 (1992); H. F. Chau and K. S. Cheng, Phys. Rev. E **47**, 2394 (1993).
[6] A. Gabrielov, Physica A **195**, 253 (1993).
[7] H. F. Chau and K. S. Cheng, J. Math. Phys. **34**, 5109 (1993).
[8] S. K. Chung and H. F. Chau, J. Math. Phys. **34**, 4014 (1993).
[9] E. R. Speer, J. Stat. Phys. **71**, 61 (1993).
[10] H. F. Chau and K. S. Cheng, Phys. Lett. A **170**, 195 (1992).